Notes for *Elementary Differential Geometry* by O'Neill

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Ch1: Calculus on Euclidean Space

1.3: Directional Derivatives

Definition 0.1. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable and $\mathbf{v}_p$ be a tangent vector to $\mathbb{R}^3$, then the directional derivative of f with respect to $\mathbf{v}_p$ is defined as*

$$\mathbf{v}_p[f] = \frac{d}{dt}(f(\mathbf{p} + t\mathbf{v}))\Big|_{t=0}$$

Lemma 0.1. *If $\mathbf{v}_p = (v_1, v_2, v_3) \in T_p\mathbb{R}^3$, then*

$$\mathbf{v}_p[f] = \sum_i v_i \frac{\partial f}{\partial x_i}(\mathbf{p})$$

Theorem 0.2. *Let f and g be functions on $\mathbb{R}^3$, $\mathbf{v}_p$ and $\mathbf{w}_p$ be tangent vectors, and a and b numbers. Then*

1. $(a\mathbf{v}_p + b\mathbf{w}_p)[f] = a\mathbf{v}_p[f] + b\mathbf{w}_p[f]$.
2. $\mathbf{v}_p[af + bg] = a\mathbf{v}_p[f] + b\mathbf{v}_p[g]$.
3. $\mathbf{v}_p[fg] = \mathbf{v}_p[f] \cdot g(\mathbf{p}) + f(\mathbf{p}) \cdot \mathbf{v}_p[g]$.

1.4: Curves in $\mathbb{R}^3$

Definition 0.3. *A **curve** in $\mathbb{R}^3$ is a differentiable function $\alpha : I \to \mathbb{R}^3$ from an open interval I into $\mathbb{R}^3$.*

Definition 0.4. *Let $\alpha : I \to \mathbb{R}^3$ be a curve. If $h : J \to I$ is a differentiable function on an open interval J, then the composite function*

$$\beta = \alpha(h) : J \to \mathbb{R}^3$$

*is a curve called a **reparametrization** of α by h.*

Lemma 0.2. *If β is a reparametrization of α by h, then*

$$\beta'(s) = \left(\frac{dh}{ds}\right)(s)\alpha'(h(s))$$

Lemma 0.3. *Let α be a curve in $\mathbb{R}^3$ and let f be a differentiable function on $\mathbb{R}^3$, then*

$$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t)$$

1.5: 1-forms

Definition 0.5. *A* ***1-form*** $\phi : T_p\mathbb{R}^3$ *is a function on the set of all tangent vectors to* $\mathbb{R}^3$ *such that* ϕ *is linear at each point. I.e.*

$$\phi(a\mathbf{v} + b\mathbf{w}) = a\phi(\mathbf{v}) + b\phi(\mathbf{w})$$

Definition 0.6. *If* $f : \mathbb{R}^3 \to \mathbb{R}$ *is differentiable then the* ***differential*** df *of* f *is the 1-form such that*

$$df(\mathbf{v}_p) = \mathbf{v}_p[f], \quad \forall \mathbf{v}_p \in T_p\mathbb{R}^3$$

Lemma 0.4. *If* f *is a differentiable function on* $\mathbb{R}^3$*, then*

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

Lemma 0.5. *Let* $f : \mathbb{R}^3 \to \mathbb{R}$ *and* $h : \mathbb{R} \to \mathbb{R}$ *be differentiable functions. Then the composite function* $h(f) : \mathbb{R}^3 \to \mathbb{R}$ *is also differentiable. Then*

$$d(h(f)) = h'(f)df$$

1.6: Differential Forms

Multiplication for differential forms is antisymmetric. So

$$dx_i dx_j = -dx_j dx_i$$

which implies that $dx_i dx_i = 0$. Now note that

- A 0-form is a differentiable function f.
- A 1-form is an expression $fdx + gdy + hdz$.
- A 2-form is an expression $fdxdy + gdxdz + hdydz$.
- A 3-form is an expression $fdxdydz$.

Lemma 0.6. *If* ϕ *and* ψ *are 1-forms, then*

$$\phi \wedge \psi = -\psi \wedge \phi$$

Definition 0.7. *If* $\phi = \sum f_i dx_i$ *is a 1-form on* $\mathbb{R}^3$*, the* ***exterior derivative*** *of* ϕ *is the 2-form* $d\phi = \sum df_i \wedge dx_i$.

Theorem 0.8. *Let* f *and* g *be functions,* ϕ *and* ψ *be 1-forms. Then*

1. $d(fg) = dfg + fdg$.
2. $d(f\phi) = df \wedge \phi + fd\phi$.
3. $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$.

Ch2: Frame Fields

2.2: Curves

Theorem 0.9. *If* α *is a regular curve in* $\mathbb{R}^3$*, then there exists a reparametrization* β *of* α *such that* β *is unit-speed.*

2.3: The Frenet Formulas

Theorem 0.10. ***(Frenet formulas)**: If $\beta : I \to \mathbb{R}^3$ is a unit-speed curve with curvature $\kappa > 0$, then*

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

Corollary 0.11. *Let β be a unit-speed curve in $\mathbb{R}^3$ with $\kappa > 0$. Then β is a plane curve if and only if $\tau = 0$.*

Lemma 0.7. *If β is a unit-speed curve with constant curvature $\kappa > 0$ and $\tau = 0$, then β is part of a circle with radius $1/\kappa$.*

2.4: Arbitrary Speed Curves

Theorem 0.12. ***(Frenet formulas)**: If $\alpha : I \to \mathbb{R}^3$ is a regular curve with curvature $\kappa > 0$, then*

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa v & 0 \\ -\kappa v & 0 & \tau v \\ 0 & -\tau v & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

Theorem 0.13. *Let α be a regular curve in $\mathbb{R}^3$. Then*

$$T = \frac{\alpha'}{\|\alpha'\|} \qquad N = B \times T \qquad B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}$$

and

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} \qquad \tau = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\|\alpha' \times \alpha''\|^2}$$

Definition 0.14. *A regular curve α in $\mathbb{R}^3$ is a **cylindrical helix** if T has constant angle θ with some fixed unit vector $\mathbf{u}$. I.e. $T(t) \cdot \mathbf{u} = \cos\theta$, $\forall t$.*

Theorem 0.15. *A regular curve α with $\kappa > 0$ is a cylindrical helix if and only if the ratio τ/κ is constant.*

2.5: Covariant Derivatives

Definition 0.16. *Let W be a vector field on $\mathbb{R}^3$ and $\mathbf{v}$ be a tangent vector field to $\mathbb{R}^3$ at point $\mathbf{p}$. Then the **covariant derivative** of W with respect to $\mathbf{v}$ is*

$$\nabla_v W = W(\mathbf{p} + t\mathbf{v})'(0)$$

Lemma 0.8. *If $W = (w_1, w_2, w_3)$ is a vector field on $\mathbb{R}^3$ and $\mathbf{v}$ is a tangent vector at ı then*

$$\nabla_v W = (\mathbf{v}[w_1], \mathbf{v}[w_2], \mathbf{v}[w_3])$$

Theorem 0.17. *Let $\mathbf{v}$ and $\mathbf{w}$ be tangent vectors to $\mathbb{R}^3$ at ı and let Y and Z be vector fields on $\mathbb{R}^3$. Then for numbers a, b and functions f,*

1. $\nabla_{av+bw} Y = a\nabla_v Y + b\nabla_w Y$.
2. $\nabla_v(aY + bZ) = a\nabla_v Y + b\nabla_v Z$.
3. $\nabla_v(fY) = \mathbf{v}[f]Y(\mathbf{p}) + f(\mathbf{p})\nabla_v Y$.
4. $\mathbf{v}[Y \cdot Z] = \nabla_v Y \cdot Z(\mathbf{p}) + Y(\mathbf{p}) \cdot \nabla_v Z$.

Ch3: Euclidean Geometry

3.5: Congruence of Curves

Definition 0.18. *Two curves $\alpha, \beta : I \to \mathbb{R}^3$ are **congruent** if there exists an isometry F of $\mathbb{R}^3$ such that $\beta = F(\alpha)$.*

Theorem 0.19. *If $\alpha, \beta : I \to \mathbb{R}^3$ are unit-speed curves such that $\kappa_\alpha = \kappa_\beta$ and $\tau_\alpha = \pm\tau_\beta$, then α and β are congruent.*

Corollary 0.20. *Let α be a unit-speed curve in $\mathbb{R}^3$. Then α is a helix if and only if both its curvature and torsion are nonzero constants.*

Corollary 0.21. *Let $\alpha, \beta : I \to \mathbb{R}^3$ be arbitrary-speed curves. If*

$$v_\alpha = v_\beta > 0 \qquad \kappa_\alpha = \kappa_\beta > 0 \qquad \tau_\alpha = \pm\tau_\beta$$

then the curves α and β are congruent.

Ch4: Calculus on a Surface

4.1: Surfaces in $\mathbb{R}^3$

Recall that a mapping is a function whose coordinates are differentiable.

Definition 0.22. *A **coordinate patch** $\mathbf{x} : D \to \mathbb{R}^3$ is a one-to-one regular mapping of an open set D of $\mathbb{R}^2$ into $\mathbb{R}^3$.*

Remark 0.23. *Regularity of a mapping can be checked by ensuring that $\mathbf{x}_u \times \mathbf{x}_v \neq 0$ everywhere.*

Definition 0.24. *A **surface** in $\mathbb{R}^3$ is a subset M of $\mathbb{R}^3$ such that for each point $\mathbf{p} \in M$, there exists a proper patch in M whose image contains a neighborhood of $\mathbf{p}$.*

Theorem 0.25. *Let g be a differentiable real-valued function on $\mathbb{R}^3$ and c a number. Then $M : g(x, y, z) = c$ is a surface if the $dg \neq 0$ at every point.*

4.2: Patch Computations

Definition 0.26. *Denote by $\mathbf{x}_u$ and $\mathbf{x}_v$ the respective partial derivatives (velocities) of the u and v parameter curves.*

Definition 0.27. *A regular mapping $\mathbf{x} : D \to \mathbb{R}^3$ whose image lies in a surface M is called a **parametrization** of the region $\mathbf{x}(D)$ in M.*

So when we relax the one-to-one condition on a coordinate patch, we get a parametrization.

Definition 0.28. *A **ruled surface** is a surface swept out by a straight line L moving along a curve β. The various positions of the generating line L are called the **rulings** of the surface. Such a surface always has a **ruled parametrization***

$$\mathbf{x}(u, v) = \beta(u) + v\delta(u)$$

where δ points along L.

4.3: Differentiable Functions and Tangent Vectors

Definition 0.29. *A function $f : M \to \mathbb{R}$ is* ***differentiable*** *if for any coordinate patch* $\mathbf{x}$*, $f(\mathbf{x}) : D \to \mathbb{R}$ is differentiable in the usual Euclidean sense. Likewise we extend this to functions $F : M \to \mathbb{R}^n$.*

Lemma 0.9. *If $\alpha : I \to M$ is a curve whose route lies in the image $\mathbf{x}(D)$ of a single patch* $\mathbf{x}$*, then there exist unique differentiable functions a_1, a_2 on I such that*

$$\alpha(t) = \mathbf{x}(a_1(t), a_2(t)), \quad \forall t \in I$$

Theorem 0.30. *If $M \subset \mathbb{R}^3$ is a surface and $F : \mathbb{R}^n \to \mathbb{R}^3$ is a differentiable mapping whose image lies in M, then considered as a mapping $F : \mathbb{R}^n \to M$ into M, F is differentiable.*

Corollary 0.31. ***(Smooth overlap)*** *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are patches in a surface $M \subset \mathbb{R}^3$ whose images overlap then the composite functions $\mathbf{x}^{-1}\mathbf{y}$ and $\mathbf{y}^{-1}\mathbf{x}$ are (differentiable) mappings defined on open sets of $\mathbb{R}^2$.*

Corollary 0.32. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are overlapping patches in M, then there exist unique differentiable functions $\overline{u}$ and $\overline{v}$ such that*

$$\mathbf{y}(u, v) = \mathbf{x}(\overline{u}(u, v), \overline{v}(u, v))$$

Definition 0.33. *Let $\mathbf{p} \in M \subset \mathbb{R}^3$. A tangent vector* $\mathbf{v}$ *in $\mathbb{R}^3$ is* ***tangent*** *to M at* $\mathbf{p}$ *if* $\mathbf{v}$ *is a velocity vector of some curve in M.*

The set of all tangent vectors to M at a point $\mathbf{p} \in M$ will be denoted T_pM.

Lemma 0.10. *If $\mathbf{p} \in M$ and $\mathbf{x}(u_0, v_0) = \mathbf{p}$, then any $\mathbf{v} \in T_pM$ can be written as a linear combination of $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$.*

Lemma 0.11. *If $M : g = c$ is a surface in $\mathbb{R}^3$, then the gradient vector field, ∇g, is a non-vanishing normal vector field on M.*

Definition 0.34. *Let $\mathbf{v} \in T_pM$ and $f : M \to \mathbb{R}$ be differentiable. Then define*

$$\mathbf{v}[f] = \frac{d}{dt}(f\alpha)(0)$$

for all curves α in M with initial velocity $\mathbf{v}$*.*

4.4: Differential Forms on a Surface

Definition 0.35. *A* ***2-form*** *$\eta : T_pM \times T_pM \to \mathbb{R}$ on a surface M is a real-valued function on all ordered pairs of tangent vectors on M such that*

1. *$\eta(\mathbf{v}, \mathbf{w})$ is linear in* $\mathbf{v}$ *and* $\mathbf{w}$*.*
2. *$\eta(\mathbf{v}, \mathbf{w}) = -\eta(\mathbf{w}, \mathbf{v})$.*

Note that this definition implies that $\eta(\mathbf{v}, \mathbf{v}) = 0$, $\forall \mathbf{v} \in T_pM$.

Definition 0.36. *If ϕ and ψ are 1-forms on a surface M, the* ***wedge product*** *$\phi \wedge \psi$ is the 2-form on M such that*

$$(\phi \wedge \psi)(\mathbf{v}, \mathbf{w}) = \phi(\mathbf{v})\psi(\mathbf{w}) - \phi(\mathbf{w})\psi(\mathbf{v}), \qquad \forall \mathbf{v}, \mathbf{w} \in T_pM$$

In general, if ξ is a p-form and η is a q-form,

$$\xi \wedge \eta = (-1)^{pq}\eta \wedge \xi$$

Definition 0.37. *Let ϕ be a 1-form on M. Then the **exterior derivative** $d\phi$ of ϕ is the 2-form such that for any patch $\mathbf{x}$ in M,*

$$d\phi(\mathbf{x}_u, \mathbf{x}_v) = \frac{\partial}{\partial u}(\phi(\mathbf{x}_v)) - \frac{\partial}{\partial v}(\phi(\mathbf{x}_u))$$

It can be shown that this definition agrees on overlaps between patches.

Theorem 0.38. *If $f : M \to \mathbb{R}$ is a function, then $d(df) = 0$.*

Definition 0.39. *A differential form ϕ is **closed** if $d\phi = 0$.*

Definition 0.40. *A differential form ϕ is **exact** if $\phi = d\xi$ for some differential form ξ.*

4.5: Mappings of Surfaces

Definition 0.41. *A function $F : M \to N$ between surfaces is **differentiable** if for each patch $\mathbf{x}$ in M and $\mathbf{y}$ in N, $\mathbf{y}^{-1}F\mathbf{x}$ is differentiable in the Euclidean sense. F is then called a **mapping of surfaces**.*

Definition 0.42. *Let $F : M \to N$ be a mapping of surfaces. Then the **tangent map** $F_* : T_pM \to T_{F(p)}N$ of F is defined such that if α is a curve in M with $\alpha'(0) = \mathbf{v} \in T_pM$, then $F_*(\mathbf{v}) = F(\alpha)'(0)$.*

Definition 0.43. *A mapping $F : M \to N$ with a inverse is called a **diffeomorphism**.*

Theorem 0.44. *Let $F : M \to N$ be a mapping of surfaces and suppose that $F_{*p} : T_p(M) \to T_{F(p)}N$ is a linear isomorphism at some point $\mathbf{p} \in M$. Then there exists a neighborhood $\mathcal{U}$ of $\mathbf{p}$ such that the restriction of F to $\mathcal{U}$ is a diffeomorphism onto a neighborhood $\mathcal{V}$ of $F(\mathbf{p}) \in M$.*

This implies that a one-to-one regular mapping F of M onto N is a diffeomorphism.

When there is a diffeomorphism between two surfaces, we say that they are *diffeomorphic*.

Definition 0.45. *Let $F : M \to N$ be a mapping of surfaces.*

1. *If ϕ is a 1-form on N, let $F^*\phi$ be the 1-form on M such that*

$$(F^*\phi)(\mathbf{v}) = \phi(F_*\mathbf{v}), \quad \forall \mathbf{v} \in T_pM$$

2. *If η is a 2-form on N, let $F^*\eta$ be the 2-form on M such that*

$$(F^*\eta)(\mathbf{v}, \mathbf{w}) = \eta(F_*\mathbf{v}, F_*\mathbf{w}), \quad \forall \mathbf{v}, \mathbf{w} \in T_pM$$

Theorem 0.46. *Let $F : M \to N$ be a mapping of surfaces and let ξ and η be forms on N. Then*

1. *$F^*(\xi + \eta) = F^*\xi + F^*\eta$.*
2. *$F^*(\xi \wedge \eta) = F^*\xi \wedge F^*\eta$.*
3. *$F^*(d\xi) = d(F^*\xi)$.*

4.6: Integration of Forms

Definition 0.47. *Let ϕ be a 1-form on M and let $\alpha : [a,b] \to M$ be a curve segment in M. Then define*

$$\int_\alpha \phi = \int_{[a,b]} \alpha^* \phi = \int_a^b \phi(\alpha'(t))dt$$

Theorem 0.48. *Let f be a function on M and let $\alpha : [a,b] \to M$ be a curve segment in M from $\mathbf{p} = \alpha(a)$ to $\mathbf{q} = \alpha(b)$. Then*

$$\int_\alpha df = f(\mathbf{q}) - f(\mathbf{p})$$

Importantly, the result above does not depend on the path chosen from $\mathbf{p}$ to $\mathbf{q}$.

Definition 0.49. *Let η be a 2-form on M and let $\mathbf{x} : R \to M$ be a 2-segment in M. Then define*

$$\iint_\mathbf{x} \eta = \iint_R \mathbf{x}^* \eta = \int_a^b \int_c^d \eta(\mathbf{x}_u, \mathbf{x}_v) du dv$$

Definition 0.50. *Let $\mathbf{x} : R \to M$ be a 2-segment in M with R the closed rectangle $a \leqslant u \leqslant b, c \leqslant v \leqslant d$. The **edge curves** of $\mathbf{x}$ are the curve segments $\alpha, \beta, \gamma, \delta$ such that*

$$\alpha(u) = \mathbf{x}(u,c)$$

$$\beta(v) = \mathbf{x}(b,v)$$

$$\gamma(u) = \mathbf{x}(u,d)$$

$$\delta(v) = \mathbf{x}(a,v)$$

Definition 0.51. *The **boundary** $\partial\mathbf{x}$ of the 2-segment $\mathbf{x}$ is*

$$\partial\mathbf{x} = \alpha + \beta - \gamma - \delta$$

Theorem 0.52. ***(Stokes' Theorem)*** *If ϕ is a 1-form on M and $\mathbf{x} : R \to M$ is a 2-segment, then*

$$\iint_\mathbf{x} d\phi = \int_{\partial\mathbf{x}} \phi$$

Lemma 0.12. *Let $\alpha(h) : [a,b] \to M$ be a reparametrization of a curve segment $\alpha : [c,d] \to M$ by $h : [a,b] \to [c,d]$. For any 1-form ϕ on M,*

1. *If h is orientation-preserving, i.e. $h(a) = c$ and $h(b) = d$, then*

$$\int_{\alpha(h)} \phi = \int_\alpha \phi$$

2. *If h is orientation-reversing, i.e. $h(a) = d$ and $h(b) = c$, then*

$$\int_{\alpha(h)} \phi = -\int_\alpha \phi$$

4.7: Topological Properties of Surfaces

Definition 0.53. *A surface is **connected** if $\forall \mathbf{p}, \mathbf{q} \in M$, there exists a curve segment in M from $\mathbf{p}$ to $\mathbf{q}$.*

Lemma 0.13. *A surface M is compact if and only if it can be covered by the images of a finite number of 2-segments.*

Lemma 0.14. *A continuous function f on a compact region $\mathcal{R}$ in a surface M takes on a maximum at some point of M.*

Definition 0.54. *A surface M is **orientable** if there exists a differentiable (or merely continuous) 2-form μ on M that is nonzero at every point of M.*

Theorem 0.55. *A surface $M \subset \mathbb{R}^3$ is orientable if and only if there exists a unit normal vector field on M. If M is connected and orientable, there are exactly two unit normals, $\pm U$.*

Definition 0.56. *A closed curve α in M is **homotopic to a constant** if there is a 2-segment $\mathbf{x} : R \to M$ (called a **homotopy**) defined on $R : a \leqslant u \leqslant b, 0 \leqslant v \leqslant 1$ such that α is the base curve of $\mathbf{x}$ and the other three edge curves are constant at $\mathbf{p} = \alpha(a) = \alpha(b)$.*

The way to think a curve being homotopic to a constant is that when $v = 0$, we get the base curve $\alpha(u)$. But as we increase v, the curve shrinks, maintaining the endpoints $\alpha(a) = \alpha(b) = \mathbf{p}$. When we set $v = 1$, the curve is constant at $\alpha(u) = \mathbf{p}$.

Definition 0.57. *A surface M is **simply connected** if it is connected and every loop in M is homotopic to a constant.*

Here a *loop* is a curve such that $\alpha(a) = \alpha(b) = \mathbf{p}$ but not necessarily $\alpha'(a) = \alpha'(b)$.

Lemma 0.15. *Let ϕ be a closed 1-form on a surface M. If a loop α in M is homotopic to a constant, then*

$$\int_\alpha \phi = 0$$

Lemma 0.16. ***(Poincare)*** *On a simply connected surface, every closed 1-form is exact.*

Theorem 0.58. *A compact surface in $\mathbb{R}^3$ is orientable.*

Theorem 0.59. *A simply connected surface is orientable.*

4.8: Manifolds

Now we construct surfaces without an embedding space. We will use the notion of an *abstract patch* which is simply a one-to-one function from an open set $D \subset \mathbb{R}^2$ into M.

Definition 0.60. *A **surface** is a set M with a collection $\mathcal{P}$ of abstract patches satisfying:*

1. ***The covering axiom:*** *The images of the patches in the collection $\mathcal{P}$ cover M.*
2. ***The smooth overlap axiom:*** *$\forall \mathbf{x}, \mathbf{y} \in \mathcal{P}$, the composite functions, $\mathbf{y}^{-1}\mathbf{x}$ and $\mathbf{x}^{-1}\mathbf{y}$ are Euclidean differentiable and defined on open sets of $\mathbb{R}^2$.*
3. ***Hausdorff axiom:*** *$\forall \mathbf{p}, \mathbf{q} \in M$ with $\mathbf{p} \neq \mathbf{q}$, there exist disjoint (non-overlapping) patches $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{p} \in \mathbf{x}(D)$ and $\mathbf{q} \in \mathbf{y}(E)$.*

Definition 0.61. *Let $\alpha : I \to M$ be a curve in an abstract surface M. For each $t \in I$, the **velocity vector** $\alpha'(t)$ is defined such that*

$$\alpha'(t)[f] = \frac{d(f\alpha)}{dt}(t)$$

for every differentiable $f : M \to \mathbb{R}$.

Definition 0.62. *An **n-dimensional manifold** M is an abstract surface where the abstract patches map from $D \to M$ where D is an open subset of $\mathbb{R}^n$.*

Ch5: Shape Operators

Two surfaces in $\mathbb{R}^3$ have the same shape (i.e. "same" shape operator) iff they are congruent.

Note that in this chapter we assume that $M \subset \mathbb{R}^3$ is connected and regular.

5.1: The Shape Operator of $M \subset \mathbb{R}^3$

Definition 0.63. *Let Z be a vector field on M and $\mathbf{v} \in T_pM$. Define the* ***covariant derivative****, $\nabla_v Z$ as*

(i) Let α be a curve in M with $\alpha(0) = \mathbf{p}$, $\alpha'(0) = \mathbf{v} \in T_pM$. Then $Z(\alpha(t))$ is a vector field in α and we define

$$\nabla_v Z = \frac{d}{dt} Z(\alpha(t))\bigg|_{t=0}$$

(ii) Write $Z = (Z_1, Z_2, Z_3)$ and define

$$\nabla_v Z = \Big(v[Z_1], v[Z_2], v[Z_3]\Big)$$

Definition 0.64. *For $\mathbf{p} \in M$ and $\mathbf{v} \in T_pM$ we define the* ***shape operator*** *to be $S_p(\mathbf{v}) = -\nabla_v U$.*

Lemma 0.17. *The shape operator is symmetric,*

$$S(\mathbf{v}) \cdot \mathbf{w} = S(\mathbf{w}) \cdot \mathbf{v}, \quad \forall k\mathbf{v}, \mathbf{w} \in T_pM$$

5.2: Normal Curvature

Lemma 0.18. *If α is a curve on M, then $\alpha'' \cdot U = S(\alpha') \cdot \alpha'$.*

Definition 0.65. *Let $u \in T_pM$ be a unit tangent vector. The* ***normal curvature*** *of M in the u-direction is $k(u) = S(u) \cdot u$.*

Remark 0.66.

1. *If $k(\mathbf{u}) > 0$, then $N(0) = U(\mathbf{p})$ so the surface M is bending toward $U(\mathbf{p})$ in the $\mathbf{u}$ direction.*
2. *If $k(\mathbf{u}) < 0$, then $N(0) = -U(\mathbf{p})$, so the surface M is bending away from $U(\mathbf{p})$ in the $\mathbf{u}$ direction.*
3. *If $k(\mathbf{u}) = 0$, then the rate of bending is unusually small.*

Definition 0.67. *The max, k_1, and min, k_2 of the normal curvature are called the* ***principal curvatures*** *of M at p. The corresponding directions are called the* ***principal vectors / direction****.*

Definition 0.68. *A point p is* ***umbilic*** *if $k_1 = k_2$ at p.*

Theorem 0.69. *(i) If $k_1 = k_2$, then $S = k_1 Id$ at p.*

(ii) If $k_1 > k_2$, then there exist exactly two principal directions. Furthermore, these are eigenvectors of S with $S(u_1) = k_1 u_1$ and $S(u_2) = k_2 u_2$.

Remark 0.70. *Locally and after translation and rotation, $M \subset \mathbb{R}^3$ may be approximated as $z = f(x, y)$, where $f_x(0)$ and $f_y(0)$ correspond to principle directions at $f(0, 0)$. In terms of the principle curvatures, we may write a quadratic approximation as*

$$z = \frac{1}{2}(k_1 x^2 + k_2 y^2)$$

Definition 0.71. *D is a **derivation** on an $\mathbb{R}$-algebra A if it is an operation $D : A \to A$ such that*

(i) $D(af + bg) = aD(f) + bD(g)$.

(ii) $D(fg) = D(f)g + fD(g), \forall f, g \in A$.

5.3: Gaussian Curvature

Definition 0.72. *The **Gaussian curvature** of $M \subset \mathbb{R}^3$ is the real-valued function $K = \det S = M$.*

Definition 0.73. *The **mean curvature** of $M \subset \mathbb{R}^3$ is $H = \frac{1}{2} tr\, S$.*

Remark 0.74. *With respect to principal vectors e_1, e_2,*

$$S = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \quad K = k_1 k_2 \quad H = \frac{1}{2}(k_1 + k_2)$$

Remark 0.75. 1. *If $K(\mathbf{p}) > 0$, then M is bending away from its tangent plane in all tangent directions at $\mathbf{p}$ and thus M locally looks like a paraboloid.*

2. *If $K(\mathbf{p}) < 0$, then M is locally saddle shaped near $\mathbf{p}$.*

Lemma 0.19. *If $\mathbf{v}, \mathbf{w} \in T_p(M)$ are linearly independent, then*

$$S(\mathbf{v}) \times S(\mathbf{w}) = K(\mathbf{p})\mathbf{v} \times \mathbf{w}$$

$$S(\mathbf{v}) \times \mathbf{w} + \mathbf{v} \times S(\mathbf{w}) = 2H(\mathbf{p})\mathbf{v} \times \mathbf{w}$$

Lemma 0.20. *In an oriented region of M, $k_1, k_2 = H \pm \sqrt{H^2 - K}$. Thus k_1 and k_2 are continuous in this region but need not be differentiable depending on if $H^2 - K = 0$ (if region contains umbilic points).*

Remark 0.76. *k_1, k_2 smooth away from umbilic points.*

Definition 0.77. *If $K = 0$ we say M is **flat**.*

Definition 0.78. *If $H = 0$ we say M is **minimal**.*

5.4: Computational Techniques

Definition 0.79. *Let $\mathbf{x} : D \to M$ be a coordinate patch. Then define the real-valued functions,*

$$E = \mathbf{x}_u \cdot \mathbf{x}_u, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v$$

$$l = S(\mathbf{x}_u) \cdot \mathbf{x}_u = U \cdot \mathbf{x}_{uu}, \quad m = S(\mathbf{x}_u) \cdot \mathbf{x}_v = U \cdot \mathbf{x}_{uv}, \quad n = S(\mathbf{x}_v) \cdot \mathbf{x}_v = U \cdot \mathbf{x}_{vv}$$

Definition 0.80. *Let $\mathbf{v} = v_1\mathbf{x}_u + v_2\mathbf{x}_v$ and $\mathbf{w} = w_1\mathbf{x}_u + w_2\mathbf{x}_v$. Define the **first fundamental form** as*

$$\mathbf{v} \cdot \mathbf{w} = Ev_1w_1 + F(v_1w_1 + v_2w_2) + Gv_2w_2$$

Theorem 0.81.

$$K = \frac{nl - m^2}{EG - F^2} \qquad H = \frac{Gl + En - 2Fm}{2(EG - F^2)}$$

5.5: The Implicit Case

Lemma 0.21. *Let V, W be two tangent vector fieldds on M such that $V \times W = Z$. Then*

$$K = \frac{Z \cdot (\nabla_v Z \times \nabla_w Z)}{\|Z\|^4} \qquad H = -\frac{Z \cdot ((\nabla_v Z) \times W + V \times (\nabla_W Z))}{2\|Z\|^3}$$

5.6: Special Curves in Surfaces

Definition 0.82. *A curve $\alpha(t)$ is a **line of curvature** or **principle curve** if $\alpha'(t)$ is a principal vector for all t.*

Lemma 0.22. *Let α be a regular curve in $M \subset \mathbb{R}^3$ and U be a unit normal vector field restricted to α. Then*

1. *α is principle if and only if U' and α' are collinear at each point.*
2. *If α is a principle curve, then the principle curvature of M in the direction of α' is $(\alpha'' \cdot U)/(\alpha' \cdot \alpha')$.*

Lemma 0.23. *Let α be a curve cut from a surface $M \subset \mathbb{R}^3$ by a plane P. If the angle between M and P is constant along α, then α is a principle curve of M.*

Theorem 0.83. *For a surface of revolution, the principal directions are given by $\mathbf{x}_u/\|\mathbf{x}_u\|$ and $\mathbf{x}_v/\|\mathbf{x}_v\|$.*

Definition 0.84. *A curve $\alpha(t)$ is **asymptotic** if its normal curvature is everywhere zero.*

Lemma 0.24. *Let $\mathbf{p} \in M \subset \mathbb{R}^3$.*

1. *If $K(\mathbf{p}) > 0$, then there are no asymptotic directions at $\mathbf{p}$.*
2. *If $K(\mathbf{p}) < 0$, then there are exactly two asymptotic directions at $\mathbf{p}$ and they are bisected by the principle directions at angle θ such that*

$$\tan^2 \theta = \frac{-k_1(\mathbf{p})}{k_2(\mathbf{p})}$$

3. *If $K(\mathbf{p}) = 0$, then every direction is asymptotic if $\mathbf{p}$ is a planar point. Otherwise there is exactly one asymptotic direction and it is also principle.*

Lemma 0.25. *A ruled surface M has $K \leqslant 0$. $K = 0$ if and only if unit normal U is parallel along each ruling of M.*

Definition 0.85. *A curve $\alpha \subset M \subset \mathbb{R}^3$ is a **geodesic** if α'' is always normal to M.*

Definition 0.86. *A **closed geodesic** is a geodesic segment $\alpha : [a, b] \to M$ that is smoothly closed, i.e. $\alpha'(a) = \alpha'(b)$, and thus may be extended by periodicity to the whole real line.*

Remark 0.87. *On a surface of revolution, all meridians are geodesics.*

5.7: Surfaces of Revolution

Definition 0.88. *Given a profile curve $\alpha(u) = (f(u), 0, g(u))$ with $\|\alpha'\|^2 > 0$ and $f > 0$, we can parametrize a surface of revolution as*

$$X(u, v) = \Big(f(u)\cos v, f(u)\sin v, g(u)\Big)$$

Theorem 0.89. *If a surface of revolution is minimal, then M is contained in a plane or catenoid.*

Lemma 0.26. *For a canonical parametrization (unit-speed) of a surface of revolution,*

$$E = 1, \quad F = 0, \quad G = f^2$$

and

$$K = -\frac{f''}{f}$$

Ch6: Geometry of Surfaces in $\mathbb{R}^3$

Key question: How does the shape of a surface affect its other properties?

6.1: The Fundamental Equations

Definition 0.90. *A **euclidean frame field** on $M \subset \mathbb{R}^3$ consists of three vector fields E_1, E_2, E_3 on M that are orthonormal at each point.*

Definition 0.91. *If $E_3 = U$ is normal to M then we call this an **adapted frame field**.*

Lemma 0.27. *There exists an adapted frame field on M iff M is orientable and exists a non-vanishing (tangent) vector field V on M.*

Definition 0.92. *The **connection one-forms** ω_{ij} of an adapted frame field are those such that for $\mathbf{v} \in T_pM$,*

$$\nabla_v E_i = \sum_{i=1}^{3} \omega_{ij}(\mathbf{v}) E_j(p)$$

So $\omega_{ij}(v) = (\nabla_v E_i) \cdot E_j(p)$.

Write

$$\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$$

Theorem 0.93. *$w = (dA)A^t$ which implies*

$$\omega_{ij} = \sum_{k=1}^{3} (da_{ik})(a^t)_{kj} = \sum_{k=1}^{3} da_{ik} a_{jk}$$

Definition 0.94. *The **dual one-forms** of E_1, E_2, E_3 are one-forms $\theta_1, \theta_2, \theta_3$ such that $\theta_i(v) = v \cdot E_i(p)$ for $v \in T_pM$. In other words,*

$$v = \sum_{i=1}^{3} \theta_i(v) E_i$$

Lemma 0.28. *If ϕ is a one-form then $\phi = \sum_i \phi(E_i)\theta_i$.*

Theorem 0.95.

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = A \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \Longrightarrow \theta_i = \sum_j a_{ij} dx_j$$

Theorem 0.96. ***Cartan Structural Equations:***

(i) The first structural equations are

$$d\theta_i = \sum_j \omega_{ij} \wedge \theta_j$$

(ii) The second structural equations are

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}$$

Proposition 0.97. *If $\{E_1, E_2, E_3\}$ is an adapted frame field for $M \subset \mathbb{R}^3$, then*

$$S(v) = \omega_{13}(v)E_1(p) + \omega_{23}(v)E_2(p)$$

Theorem 0.98. *On a surface with an adapted frame field, the structural equations become*

(i) First structural equations

$$d\theta_1 = \omega_{12} \wedge \theta_2 \qquad d\theta_2 = \omega_{21} \wedge \theta_1 = -\omega_{12} \wedge \theta_1 \qquad 0 = d\theta_3 = \omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2$$

(ii) Second structural equations

Gauss Equation: $d\omega_{12} = \omega_{13} \wedge \omega_{32}$

Codazzi Equations: $d\omega_{13} = \omega_{12} \wedge \omega_{23} \qquad d\omega_{23} = \omega_{21} \wedge \omega_{13}$

6.2: Form Computations

Lemma 0.29. *If ϕ is a one-form then $\phi = \phi(E_1)\theta_1 + \phi(E_2)\theta_2$.*

Lemma 0.30. *If μ is a two-form then $\mu = \mu(E_1, E_2)\theta_1 \wedge \theta_2$.*

Lemma 0.31. *(i) $\omega_{13} \wedge \omega_{23} = K\theta_1 \wedge \theta_2$.*

(ii) $\omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23} = 2H\theta_1 \wedge \theta_2$.

Corollary 0.99.

$$d\omega_{12} = -K\theta_1 \wedge \theta_2$$

Definition 0.100. *A* ***principal frame field*** *on $M \subset \mathbb{R}^3$ is an adapted frame field such that E_1 and E_2 are principal vectors at all points.*

Lemma 0.32. *If* **p** *is not umbilic, then there exists a principal frame field on a neighborhood of $\mathbf{p} \in M$.*

Theorem 0.101. *For principal frame fields,*

$$E_2[k_1] = (k_1 - k_2)\omega_{12}(E_1)$$

$$E_1[k_2] = (k_1 - k_2)\omega_{12}(E_2)$$

6.3: Some Global Theorems

Theorem 0.102. *If M is a connected surface with shape operator $S = 0$ then $M \subseteq$ plane.*

Lemma 0.33. *If every point of M is umbilic then $K \geqslant 0$.*

Theorem 0.103. *If every point of M is umbilic and $K > 0$ then $M \subseteq$ sphere of radius $1/\sqrt{K}$.*

Corollary 0.104. *If $M \subset \mathbb{R}^3$ compact and all-umbilic then it is an entire sphere.*

Theorem 0.105. *On every compact surface $M \subset \mathbb{R}^3$, there exists a point p with $K(p) > 0$.*

Remark 0.106. *There does not exist a compact surface with $K \leqslant 0$.*

Theorem 0.107. ***(Hilbert)** Suppose there exists a point $m \in M \subset \mathbb{R}^3$ such that*

(i) k_1 has a local max at m

(ii) k_2 has a local min at m

(iii) $k_1 > k_2$ at m (so m is not umbilic)

Then $K(m) \leqslant 0$.

Theorem 0.108. ***(Liebman)** If $M \subset \mathbb{R}^3$ is compact with K constant (and necessarily $K > 0$) then M is a sphere of radius $1/\sqrt{K}$.*

6.4: Isometries and Local Isometries

Definition 0.109. *Let $\mathbf{p}, \mathbf{q} \in M \subset \mathbf{R}^3$ and $\mathcal{C} = \{\alpha : \alpha$ is a curve segment from $\mathbf{p}$ to $\mathbf{q}\}$. Then the **intrinsic distance** $\rho(\mathbf{p}, \mathbf{q})$ is define as*

$$\rho(\mathbf{p}, \mathbf{q}) = \inf_{\alpha \in \mathcal{C}} L(\alpha)$$

where L is the length operator.

Definition 0.110. *An **isometry** $F : M \to \overline{M}$ of surfaces in $\mathbf{R}^3$ is a one-to-one mapping of M onto $\overline{M}$ that preserves dot products of tangent vectors. If F_* is the derivative map of F, then*

$$F_*(\mathbf{v}) \cdot F_*(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}, \quad \forall \mathbf{v}, \mathbf{w} \in T_pM,\ p \in M$$

Remark 0.111. *By remarks in previous chapters, an isometry F is a diffeomorphism.*

Theorem 0.112. *Isometries preserve intrinsic distance. If $F : M \to \overline{M}$ is an isometry of surfaces in $\mathbf{R}^3$,*

$$\rho(\mathbf{p}, \mathbf{q}) = \overline{\rho}(F(\mathbf{p}), F(\mathbf{q}))$$

for $\mathbf{p}, \mathbf{q} \in M$.

Remark 0.113. *If there is an isometry between two surfaces, they are said to be isometric.*

Definition 0.114. *A **local isometry** $F : M \to N$ of surfaces is a mapping that preserves dot products of tangent vectors.*

Thus an isometry is a local isometry that is one-to-one and onto. One may show that a local isometry is an isometry on a neighborhood of points.

Theorem 0.115. *Let $F : M \to N$ be a mapping and $X : D \to M$ be a patch. Let $\overline{X} = F(X) : D \to N$. Then F is a local isometry if and only if*

$$E = \overline{E}, \quad F = \overline{F}, \quad G = \overline{G}$$

Remark 0.116. *One may use this result to construct local isometries. I.e. if you have two patches, $\mathbf{x} : D \to M$ and $\mathbf{y} : D \to N$, find a function F such that $F(\mathbf{x}(u,v)) = \mathbf{y}(u,v)$ for $(u,v) \in D$ and with $E = \overline{E}, F = \overline{F}, G = \overline{G}$.*

Definition 0.117. *A mapping $F : M \to N$ is **conformal** if there exists a real-valued function $\lambda > 0$ on M such that*

$$\|F_*(\mathbf{v}_p)\| = \lambda(\mathbf{p})\|\mathbf{v}_p\|$$

*Here λ is called the **scale factor**.*

Note that a local isometry has $\lambda = 1$, so a conformal mapping can be thought of as a generalized isometry.

6.5: Intrinsic Geometry of Surfaces in $\mathbf{R}^3$

Intrinsic geometry of a surface refers to its properties which are invariant under isometry.

Lemma 0.34. *Let $F : M \to \overline{M}$ be an isometry and let E_1, E_2 be a tangent frame field on M. If $\overline{E}_1, \overline{E}_2$ is the transferred frame field on $\overline{M}$ then*

$$\theta_1 = F^*(\overline{\theta}_1), \quad \theta_2 = F^*(\overline{\theta}_2)$$

$$\omega_{12} = F^*(\overline{\omega}_{12})$$

Theorem 0.118. *(**Theorema egregium of Gauss**) Gaussian curvature is an isometric invariant. Explicitly, if $F : M \to \overline{M}$ is an isometry, then*

$$K(\mathbf{p}) = \overline{K}(F(\mathbf{p})), \forall \mathbf{p} \in M$$

6.6: Orthogonal Coordinates

Definition 0.119. *The associated frame field E_1, E_2 of an orthogonal patch ($F = 0$) $\mathbf{x} : D \to M$ consists of*

$$E_1 = \frac{\mathbf{x}_u(u,v)}{\sqrt{E(u,v)}} \qquad E_2 = \frac{\mathbf{x}_v(u,v)}{\sqrt{G(u,v)}}$$

Remark 0.120. *This yields dual one forms*

$$\theta_1 = \sqrt{E}du \qquad \theta_2 = \sqrt{G}dv$$

Proposition 0.121. *For an orthogonal patch,*

$$K = -\frac{1}{\sqrt{EG}}\left[\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u + \left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v\right]$$

6.7: Integration and Orientation

One can show that a coordinate patch $\mathbf{x} : D \to M$ distorts the area of a rectangle such that the infinitesimal area is given by $\sqrt{EG - F^2}\Delta u \Delta v$. Thus the area will be found by integrating this over certain subsets of D.

Definition 0.122. *For a rectangle $R : a \leqslant u \leqslant b, c \leqslant v \leqslant d$ with interior $R^\circ : a < u < b, c < v < d$, a 2-segment $\mathbf{x} : R \to M$ is **patchlike** is $\mathbf{x} : R^\circ \to M$ is a patch in M.*

Definition 0.123. *A **paving** of a region $\mathcal{P}$ in a surface M is a finite number of patchlike 2-segments $\mathbf{x}_1, ..., \mathbf{x}_k$ whose images fill M in such a way that each point of M is in at most one set $\mathbf{x}_i(R_i^\circ)$.*

Note that an entire compact surface is always pavable and the area of a pavable region is said to be the sum of the areas of its patchlike 2-segments.

Definition 0.124. *An **area form** on a surface M is a differentiable 2-form μ whose value on any pair of tangent vectors is*

$$\mu(\mathbf{v}, \mathbf{w}) = \pm\|\mathbf{v} \times \mathbf{w}\|$$

Lemma 0.35. *A surface M has an area form if and only if it is orientable. On a connected orientable surface there are exactly two area forms, which are negatives of each other.*

Definition 0.125. *Let v be a 2-form on a pavable oriented region $\mathcal{P}$ in a surface. The **integral of v over** $\mathcal{P}$ is*

$$\iint_{\mathcal{P}} v = \sum_i \iint_{X_i} v$$

where $X_1, ..., X_k$ is a positively oriented paving of $\mathcal{P}$.

6.8: Total Curvature

Definition 0.126. *Let K be Gaussian curvature of a compact surface M oriented by area form dM. Then*

$$\iint_M K dM$$

*is the **total Gaussian curvature** of M.*

Definition 0.127. *Let M and N be surfaces oriented by area forms dM and dN. Then the **Jacobian** of a mapping $F : M \to N$ is the real-valued function J_F on M such that*

$$F^*(dN) = J_F dM$$

Theorem 0.128. *The Gaussian curvature K of an oriented surface $M \subset \mathbb{R}^3$ is the Jacobian of its Gauss map.*

Corollary 0.129. *The total Gaussian curvature of an oriented surface $M \subset \mathbb{R}^3$ equals the algebraic area of the image of its Gauss map $G : M \to \Sigma$.*

Corollary 0.130. *Let $\mathcal{R}$ be an oriented region in $M \subset \mathbb{R}^3$ on which*

1. *The Gauss map G is one-to-one and*
2. *Either $K \geqslant 0$ or $K \leqslant 0$*

Then the total curvature of $\mathcal{R}$ is $\pm$ area of $G(\mathcal{R})$ where the sign is that of K.

Definition 0.131. *The **rotation operator** of M is the linear operator J such that*

$$J(\mathbf{v}) = U \times \mathbf{v}$$

Definition 0.132. *Let $\mathbf{v}, \mathbf{w}$ be unit tangent vectors at a point of an oriented surface M. A number ϕ is an **oriented angle** from $\mathbf{v}$ to $\mathbf{w}$ if*

$$\mathbf{w} = \cos\phi\mathbf{v} + \sin\phi J(\mathbf{v})$$

Lemma 0.36. *Let $\alpha : I \to M$ be a curve in an oriented surface M. If V and W are nonvanishing tangent vector fields on α, there is a differentiable function ϕ on I such that for each $t \in I$, $\phi(t)$ is an oriented angle from $V(t)$ to $W(t)$.*

Then any non-vanishing vector field V on M determines a positively oriented frame field,

$$E_1 = \frac{V}{\|V\|} \qquad E_2 = J(E_1) = \frac{J(V)}{\|V\|}$$

6.9: Congruence of Surfaces

Two surfaces M and $\overline{M}$ in $\mathbb{R}^3$ are **congruent** if there is an isometry F of $\mathbb{R}^3$ that carries M exactly onto $\overline{M}$.

Theorem 0.133. *If $\mathbf{F}$ is a Euclidean isometry such that $\mathbf{F}(M) = \overline{M}$, then $F = \mathbf{F}|M : M \to \overline{M}$ is an isometry. Furthermore, if M and $\overline{M}$ are suitably oriented, then F preserves shape operators,*

$$F_*(S(\mathbf{v})) = \overline{S}(F_*(\mathbf{v}))$$

Theorem 0.134. *Let M and $\overline{M}$ be oriented surfaces in $\mathbb{R}^3$. Let $F : M \to \overline{M}$ be an isometry that preserves shape operators. Then M an $d\overline{M}$ are congruent. In fact, there is a Euclidean isometry $\mathbf{F}$ such that $\mathbf{F}|M = F$.*

Ch7: Reimannian Geometry

7.1: Geometric Surfaces

Definition 0.135. *A **geometric surface** is an abstract surface M with an inner product on each tangent plane which varies smoothly.*

Definition 0.136. *A **metric tensor** g on M is a function on all ordered pairs of tangent vectors v, w at points $p \in M$ such that*

$$g_p(v, w) = \langle v, w \rangle_p$$

Remark 0.137. *The metric tensor is like a two-form but symmetric.*

Note 0.138. *Construction methods for geometric surfaces:*

1. ***Conformal Change:*** *Let $h > 0$ be differentiable on a region of $\mathbb{R}^2$, then redefine the inner product by*

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{v \cdot w}{h(\mathbf{p})^2}$$

2. ***Pullback***: *Use inner product on another geometric surface via pullback,* $F : M \to N$*,*

$$\langle \mathbf{v}, \mathbf{w} \rangle_M = \langle F_*(v), F_*(w) \rangle_N$$

3. ***Coordinate Description***: *For a coordinate patch* x *on an abstract surface* M*, defining the functions,*

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle \qquad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle \qquad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$$

defines a unique metric tensor on the image of $\mathbf{x}$*.*

Note 0.139. *As before, dual 1-forms,* θ_1, θ_2 *are uniquely determined by* $\theta_i(E_j) = \delta_{ij}$*, and connection form* ω_{12} *is uniquely determined by the first structural equations,*

$$d\theta_1 = \omega_{12} \wedge \theta_2 \qquad d\theta_2 = \omega_{21} \wedge \theta_1$$

Definition 0.140. *Let* $\{E_1, E_2\}$ *and* $\{\overline{E}_1, \overline{E}_2\}$ *be two choices of frame fields and let* φ *be the angle between* $\overline{E}_1$ *and* E_1*. If* $\overline{E}_2 = -\sin\varphi E_1 + \cos\varphi E_2$*, then the two frame fields are said to have the* ***same orientation****. If* $\overline{E}_2 = \sin\varphi E_1 - \cos\varphi E_2$ *they are said to have* ***opposite orientation****.*

Lemma 0.37. *Let* $\{E_1, E_2\}$*,* $\{\overline{E}_1, \overline{E}_2\}$ *be frame fields on the same region of* M*.*

(i) If they have the same orientation,

$$\overline{\omega}_{12} = \omega_{12} + d\varphi \qquad \overline{\theta}_1 \wedge \overline{\theta}_2 = \theta_1 \wedge \theta_2$$

(ii) If they have opposite orientation,

$$\overline{\omega}_{12} = -(\omega_{12} + d\varphi) \qquad \overline{\theta}_1 \wedge \overline{\theta}_2 = -\theta_1 \wedge \theta_2$$

Definition 0.141. *A* ***Reimannian manifold*** *is a manifold furnished with a metric.*

7.2: Gaussian Curvature

Need a new definition of Gaussian curvature now that we no longer have a shape operator.

Theorem 0.142. *On a geometric surface* M*, there is a unique real-valued function* K *such that for every frame field on* M*, the second structural equation holds,*

$$d\omega_{12} = -K\theta_1 \wedge \theta_2$$

Definition 0.143. *On a geometric surface, define the* ***Gaussian curvature*** *to be* K *such that* $d\omega_{12} = -K\theta_1 \wedge \theta_2$*.*

Corollary 0.144. *For the plane* $\mathbb{R}^2$ *with metric tensor* $\langle \mathbf{v}, \mathbf{w} \rangle = \frac{v \cdot w}{h^2(\mathbf{p})}$*, the Gaussian curvature is*

$$K = h(h_{uu} + h_{vv}) - (h_u^2 + h_v^2)$$

Proposition 0.145. *Let* $F : M \to N$ *be a regular mapping of a geometric surface* M *onto a surface* N *without geometry. Suppose that whenever* $F(\mathbf{p}_1) = F(\mathbf{p}_2)$*, there is an isometry* G_{12} *from a neighborhood of* $\mathbf{p}_1$ *to a neighborhood of* $\mathbf{p}_2$ *such that*

$$FG_{12} = F, \qquad G_{12}(\mathbf{p}_1) = \mathbf{p}_2$$

Then there is a unique metric tensor on N *that makes* F *a local isometry.*

7.3: Covariant Derivative

Lemma 0.38. *Assume there exists a covariant derivative* ∇ *on* M *which is linear, Leibnizian and such that* $\omega_{12}(V) = \langle \nabla_V E_1, E_2 \rangle$, *then* ∇ *obeys the connection equations,*

$$\nabla_V E_1 = \omega_{12}(V) E_2 \qquad \nabla_V E_2 = \omega_{12}(V) E_1$$

Furthermore, for a vector field $W = f_1 E_1 + f_2 E_2$,

$$\nabla_V W = (V[f_1] + f_2 \omega_{21}(V)) E_1 + (V[f_2] + f_1 \omega_{12}(V)) E_2$$

called the ***covariant derivative formula****.*

Theorem 0.146. *On each geometric surface* M *there exists a unique covariant derivative* ∇ *in the linear and Leibnizian properties satisfying* $\omega_{12}(V) = \langle \nabla_V E_1, E_2 \rangle$.

Definition 0.147. *A vector field* V *on a curve* α *in a geometric surface is* ***parallel*** *provided its covariant derivative vanishes,* $V' = 0$.

Note 0.148. *For* $Y = f_1 E_1 + f_2 E_2$ *along curve* α, *we write*

$$Y' = (f_1' + f_2 \omega_{21}(\alpha')) E_1 + (f_2' + f_1 \omega_{12}(\alpha')) E_2$$

Lemma 0.39. *Let* α *be a curve in a geometric surface* M, *and let* $\mathbf{v}$ *be a tangent vector at* $\mathbf{p} = \alpha(t_0)$. *Then there is a unique parallel vector field* V *on* α *such that* $V(t_0) = \mathbf{v}$.

Remark 0.149. *For a parallel vector field* V *on* α, *we say* $\alpha(t)$ *is gotten from* v *at* $\mathbf{p} = \alpha(t_0)$ *by parallel transportation along* α.

Definition 0.150. *If* $\alpha : [a, b] \to M$ *is a closed curve in the domain of a frame field,* $\varphi' + \omega_{12}(\alpha') = 0$ *leads us to define the* ***holonomy angle*** ψ_α *of* α *as*

$$\psi_\alpha = \varphi(b) - \varphi(a) = -\int_\alpha \omega_{12}$$

Lemma 0.40. *(Connection between covariant derivatives on a geometric surface and* $\mathbb{R}^3$*) If* V *and* W *are tangent vector fields on a surface* M *in* $\mathbb{R}^3$, *then*

1. $\nabla_V W$ *is the component of* $\tilde{\nabla}_V W$ *tangent to* M.
2. *If* S *is the shape operator of* M *derived from a unit normal* U, *then*

$$\tilde{\nabla}_V W = \nabla_V W + (S(V) \cdot W) U$$

7.4: Geodesics

Definition 0.151. *A curve in a geometric surface is a* ***geodesic*** *provided its acceleration is zero,* $\gamma'' = 0$.

Remark 0.152. *The velocity of a geodesic is parallel, i.e. they never turn.*

Remark 0.153. *As acceleration is preserved by isometry, geodesics are isometric invariants.*

Theorem 0.154. *Write* $\alpha'' = A_1 E_1 + A_2 E_2$ *where* A_1, A_2 *are real-valued functions. Let* $\mathbf{x}$ *be an orthogonal patch in a geometric surface* M. *A curve* $\alpha(t) = x(a_1(t), a_2(t))$ *is a geodesic of* M *iff*

$$A_1 = a_1'' + \frac{1}{2E}\left(E_u a_1'^2 + 2E_v a_1' a_2' - G_u a_2'^2 \right) = 0$$

$$A_2 = a_2'' + \frac{1}{2G}\left(-E_v a_1'^2 + 2G_u a_1' a_2' + G_v a_2'^2 \right) = 0$$

Theorem 0.155. *Given a tangent vector* $\mathbf{v}$ *to* M *at a point* $\mathbf{p}$*, there is a unique geodesic* γ *defined on an interval* I *around 0 such that* $\gamma(0) = \mathbf{p}$*and* $\gamma'(0) = \mathbf{v}$*.*

Definition 0.156. *A geometric surface is* ***complete*** *provided every maximal geodesic in* M *is defined on the whole real line* $\mathbb{R}$*.*

Lemma 0.41. *Let* E_1*,* E_2 *be a frame field and let* α *be a constant speed curve such that* α' *and* E_2 *are never orthogonal. If* $A_1 = 0$ *then* $A_2 = 0$*, hence* α *is a geodesic.*

Definition 0.157. *Let* α *be a unit-speed curve in* $M \subset \mathbb{R}^3$*,* U *be a unit normal vector field restricted to* α*, and* $V = U \times \alpha''$*. Then the* ***geodesic curvature*** κ_g *of* α *is the function such that*

$$\alpha'' = \kappa_g V + kU$$

where $k = S(T) \cdot T$ *is the normal curvature of* M *in the* T *direction.*

Corollary 0.158. *Let* β *be a unit speed curve in a region oriented by a frame field* E_1, E_2*. If* φ *is an angle function from* E_1 *to* β' *along* β*, then*

$$\kappa_g = \frac{d\varphi}{ds} + \omega_{12}(\beta')$$

Lemma 0.42. *A regular speed curve* α *in* M *is a geodesic if and only if* α *has constant speed and geodesic curvature,* $\kappa_g = 0$*.*

7.5: Clairaut Parametrizations

Definition 0.159. *A* ***Clairaut parametrization*** $\mathbf{x} : D \to M$ *is an orthogonal parametrization for which* E *and* G *depend only on* u*. I.e.* $F = 0$*,* $E_v = G_v = 0$*.*

Lemma 0.43. *If* x *is a Clairaut parametrization, then*

1. *All the* u*-parameter curves are pregeodesics.*
2. *A* v*-parameter curve* $u = u_0$ *is a geodesic iff* $G_u(u_0) = 0$*.*

Theorem 0.160. *Let* $\alpha = \mathbf{x}(a_1, a_2)$ *be a unit-speed geodesic with* $\mathbf{x}$ *a Clairaut parametrization. If* φ *is the angle from* x_u *to* α' *then the function*

$$c = G(a_1)a_2' = \sqrt{G(a_1)} \sin\varphi$$

is constant along α*. Hence* α *cannot leave the region where* $G \geqslant c^2$*.*

Definition 0.161. $c = c(\alpha)$ *from above is called the* ***slant*** *of* α *as it determines the angle* φ *at which* α *cuts across the meridians.*

Proposition 0.162. *If* $\mathbf{x}$ *is a Clairaut parametrization, then every geodesic* α *such that* α' *is never orthogonal to meridians can be parametrized as* $\beta(u) = \mathbf{x}(u, v(u))$ *where*

$$\frac{dv}{du} = \pm \frac{c\sqrt{E}}{\sqrt{G}\sqrt{G - c^2}}$$

with c *the slant of* α*. Hence by the fundamental theorem of calculus,*

$$v(u) = v(u_0) \pm \int_{u_0}^{u} \frac{c\sqrt{E}dt}{\sqrt{G}\sqrt{G - c^2}}$$

7.6: Gauss-Bonnet Theorem

Definition 0.163. *Let $\alpha : [a,b] \to M$ be a regular curve segment in an oriented geometric surface M. The **total geodesic curvature** of α is*

$$\int_\alpha \kappa_g ds = \int_{s(a)}^{s(b)} \kappa_g(s(t)) \frac{ds}{dt} dt$$

Lemma 0.44. *Let $\alpha : [a,b] \to M$ be a regular curve segment in a region of M oriented by a frame field E_1, E_2. Then*

$$\int_\alpha \kappa_g ds = \varphi(b) - \varphi(a) + \int_\alpha \omega_{12}$$

where φ is an angle function from E_1 to α' along α and ω_{12} is the connection form of E_1, E_2.

Definition 0.164. *Let $\mathbf{x} : R \to M$ be a one-to-one regular 2-segment with vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$. The **exterior angle** ε_j of $\mathbf{x}$ at $\mathbf{p}_j$ $(1 \leqslant j \leqslant 4)$ is the turning angle at $\mathbf{p}_j$ derived from the edge curves $\alpha, \beta, -\gamma, -\delta, \alpha, ...$ in order of occurrence in $\mathbf{x}$. The **interior angle** l_j at $\mathbf{p}_j$ is $\pi - \varepsilon_j$.*

Theorem 0.165. *Let $\mathbf{x} : R \to M$ be a one-to-one regular 2-segment in a geometric surface M. If dM is the area form determined by $\mathbf{x}$, then*

$$\iint_{\mathbf{x}} K dM + \int_{\partial \mathbf{x}} \kappa_g ds + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = 2\pi$$

where ε_j is the exterior angle at the vertex $\mathbf{p}_j$ of $\mathbf{x}$ $(1 \leqslant j \leqslant 4)$. This formula can be written in terms of interior angles as

$$\iint_{\mathbf{x}} K dM + \int_{\partial \mathbf{x}} \kappa_g ds = l_1 + l_2 + l_3 + l_4 - 2\pi$$

Definition 0.166. *A **rectangular decomposition** $\mathcal{D}$ of a surface M is a finite collection of one-to-one regular 2-segments $\mathbf{x}_1, ..., \mathbf{x}_f$ whose images cover M in such a way that if any two intersect, they do so in either a single common vertex or a single common edge.*

Theorem 0.167. *Every compact surface M has a rectangular decomposition.*

Theorem 0.168. *If $\mathcal{D}$ is a rectangular decomposition of a compact surface M, let v, e, and f be the number of vertices, edges and faces in $\mathcal{D}$. Then the integer $v - e + f$ is the same for every rectangular decomposition of M. This integer $\chi(M)$ is called the **Euler characteristic** of M.*

Definition 0.169. *$\Sigma[h]$ is the surface obtained by taking a sphere and adding h handles to it.*

Theorem 0.170. *If M is a compact, connected, orientable surface, there is a unique integer $h \geqslant 0$ such that M is diffeomorphic to $\Sigma[h]$.*

Corollary 0.171. *Compact orientable surfaces M and N have the same Euler characteristic iff they are diffeomorphic.*

Theorem 0.172. *(Gauss-Bonnet) The total Gaussian curvature of a compact orientable geometric surface M is 2π times its Euler characteristic:*

$$\iint_M K dM = 2\pi \chi(M)$$

Note 0.173. *This links the topology and geometry of a surface, implying that the total Gaussian curvature is a topological invariant.*

7.7: Applications of Gauss-Bonnet

Definition 0.174. *An* ***oriented polygonal region*** $\mathcal{P}$ *in a surface* M *is a (necessarily compact) oriented region furnished with a positively oriented rectangular decomposition* $\mathbf{x}_1, ..., \mathbf{x}_f$.

Definition 0.175. *A* ***boundary segment*** *of* $\mathcal{P}$ *is a curve segment* β *that is an edge curve of exactly one of the rectangles* $\mathbf{x}_i(R_i)$. *For simplicity we add the requirement that a vertex of the decomposition cannot belong to more than the boundary segments.*

Definition 0.176. *The oriented boundary* $\partial\mathcal{P}$ *of an oriented polygonal region* $\mathcal{P}$ *is the formal sum of the simple closed, oriented polygonal curves* β_i *described above:*

$$\partial\mathcal{P} = \beta_1 + \cdots + \beta_k$$

Theorem 0.177. *(Generalized Stokes' Theorem) If* ϕ *is a 1-form on an oriented polygonal region* $\mathcal{P}$, *then*

$$\iint_{\mathcal{P}} d\phi = \int_{\partial\mathcal{P}} \phi$$

In particular, if $\mathcal{P}$ *is an entire compact oriented surface* M, *then* $\iint_M d\phi = 0$.

Corollary 0.178. *The following properties of a compact orientable surfaces surface are equivalent:*

1. *There is a non-vanishing tangent vector field on* M.
2. $\chi(M) = 0$
3. M *is diffeomorphic to a torus.*

Theorem 0.179. *If* $\mathcal{P}$ *is an oriented polygonal region in a geometric surface, then*

$$\iint_{\mathcal{P}} K dM + \int_{\partial\mathcal{P}} \kappa_g ds + \sum \varepsilon_j = 2\pi\chi(\mathcal{P})$$

where $\sum \varepsilon_j$ *is the sum of the exterior angles of all the closed boundary curves comprising* $\partial\mathcal{P}$.

Corollary 0.180. *If* Δ *is a triangle in an oriented geometric surface* M, *then*

$$\iint_{\Delta} K dM + \int_{\partial\Delta} \kappa_g ds = 2\pi - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = (l_1 + l_2 + l_3) - \pi$$

Definition 0.181. *A point* $\mathbf{p}$ *is an* ***isolated singular point*** *of a vector field* V *is* V *is non-vanishing and differentiable on some neighborhood* $\mathcal{N}$ *of* $\mathbf{p}$, *except at the point* $\mathbf{p}$ *itself.*

Definition 0.182. *Let* $\alpha : [a, b] \to C$ *be a parametrization of the boundary* C *as the oriented boundary* $\partial\mathcal{D}$ *of* $\mathcal{D}$. *Let* $\varphi = \langle_\alpha(X, V)$ *be an angle function from* X_α *to* V_α *(these vector fields restricted to* α*) for some smooth vector field* X *with no singularities anywhere in* $\mathcal{D}$. *Then* $\varphi(b) - \varphi(a)$ *is called the* ***total rotation*** *and is a multiple of* 2π.

Definition 0.183. *The index of* V *at* p *is the integer*

$$ind(V, p) = \frac{\varphi(b) - \varphi(a)}{2\pi}$$

Theorem 0.184. *(Poincare-Hopf) Let* V *be a vector field on a compact oriented surface* M. *If* V *is differentiable and non-vanishing except at isolated singular points* $p_1, ..., p_k$ *then the Euler characteristic of* M *is the sum of their indices*

$$\chi(M) = \sum_{i=1}^{K} ind(V, p_i)$$